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330:542 Information Theory and Coding Spring 1997

Quiz 2

1. (*50 points*) **Source Coding**

   (a) (*25 points*) A computer generates a number X according to a known distribution $p_X(x)$ where $x \in \{1, 2, ...100\}$. The player asks a question, "Is $X = i$?" and is told either "yes," "you're too high," or "you're too low." The player continues for a total of six questions. If correct (i.e., the answer "yes" is received at some point in the sequence) the player receives $v(X)$ dollars. How should the player proceed to maximize earnings over many plays of the game? What is the value of the expected earnings? Note that $v()$ is an arbitrary real-valued function.

   (b) (*25 points*) Consider a variation of the game where now questions can be of any form desired and only "yes" or "no" answers are given. The game stops when the number is determined. That is, the game stops when by the sequence of answers the number can be uniquely determined. This does not necessarily correspond to a single yes answer. Questions cost one dollar each. What is the best strategy for the player given $v()$? What is the expected value of the earnings?

2. (*50 points*) **Discrete Channel Capacity**

   (a) (*25 points*) The "Z" channel has inputs zero and one and outputs zero and one. The transition probabilities are $p_{0|0} = 1$, $p_{1|0} = 0$ and $p_{0|1} = p_{1|1} = 1/2$. Make a sketch for reference. Find the capacity of the "Z" channel and the maximizing input probability distribution.

   (b) (*25 points*) Suppose we generate $M = 2^{nR}$ binary codewords of length n for the "Z" channel by tossing a fair coin Mn times; i.e., each n tosses constitute a codeword. This channel input distribution (equiprobable) will not achieve capacity. However, there is some maximum rate R at which we can guarantee asymptotically reliable transmission – that is, the average probability $P_e^{(n)} \to 0$ in n. Find this maximum rate R.

3. (*50 points*) **Differential Entropy and Channel Capacity**

   (a) (*25 points*) A non-negative random variable X is exponentially distributed as $f_Z(z) = a\mathbf{e}^{-az}$ with $a > 0$. Calculate the entropy of Z. Remember that the mean of an exponential distribution with parameter a is $1/a$.

   (b) (*25 points*) A memoryless channel has two possible inputs $X \in \{\alpha, \beta\}$ where $\alpha < \beta$ are real constants. The channel output is $Y = X+Z$ where Z is a non-negative exponentially distributed random variable with parameter a. If an average power constraint $E[X^2] \leq P$ is imposed on X, what is the capacity of this channel? What values of α, β and probability mass function $p_X(x)$ achieve capacity.

   HINT: This might be tricky so think broadly; i.e., there's more than one correct approach.