Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

330:542 | **Information Theory and Coding** | Spring 1996

Problem Set 6

1. Some fun with parity check codes:

   (a) Show that in a binary parity-check code, either all the code words contain an even number of ones or half the code words contain an odd number of ones and half an even number.

   (b) Let $x_{m,n}$ be the n^{th} digit in the m^{th} codeword of a parity-check code. Show that for any given n, either exactly half or all of the $x_{m,n}$ are zero. If all of the $x_{m,n}$ are zero for a given n, explain how the code could be improved.

   (c) Show that the average number of ones per code word, averaged over all code words in a parity-check code of block length N, must be at most $N/2$.

2. The Plotkin bound to be derived in this problem works for general binary codes. We will derive the Plotkin bound for linear codes.

   (a) Using problem 1c of the previous problem please show that the average weight of the nonzero code words in an (N, L) parity-check code is at most

   $$\frac{N}{2}\frac{2^L}{2^L - 1}$$

   Then use this result to show that d_{min} satisfies

   $$d_{min} \leq \frac{N}{2}\frac{2^L}{2^L - 1}$$

   (b) The above bound is effective when L is small relative to N. For larger values of L, the following bound is tighter. Show that for all j, $1 \leq j \leq L$, that

   $$d_{min} \leq \frac{N - L + j}{2}\frac{2^j}{2^j - 1} \tag{1}$$

   HINT: Consider the 2^j codewords in the code with the first $L - j$ information digits constrained to be 0. Consider that set of codewords with the first $L - j$ information digits omitted as an $(N - L + j, j)$ parity-check code and apply the bound in part 2a. *Bonus:* show that equation 1 is valid for any binary code of block length N with 2^L codewords.

   (c) Now consider N and d_{min} as fixed, $N \geq 2d_{min} - 1$. Show that the number of check digits must satisfy,

   $$N - L \geq 2(d_{min} - 1) - \lfloor \log_2 d_{min} \rfloor$$

   where $\lfloor x \rfloor$ is the largest integer not greater than x. **HINT:** Choose $j = 1 + \lfloor \log_2 d_{min} \rfloor$ and remember that $N - L$, d_{min}, and j are integers.

(d) Show that in the limit as $N \to \infty$, with fixed $\frac{d_{min}}{N} < \frac{1}{2}$, the rate in binary units, $R = L/N$ must satisfy

$$R \leq 1 - \frac{2d_{min}}{N}$$

3. Consider two parity-check codes. Code I is generated by the rule

$$\begin{aligned}
x_1 &= u_1 \\
x_2 &= u_2 \\
x_3 &= u_3 \\
x_4 &= u_1 + u_2 \\
x_5 &= u_1 + u_3 \\
x_6 &= u_2 + u_3 \\
x_7 &= u_1 + u_2 + u_3
\end{aligned}$$

using modulo-2 addition. Code II is the same except that $x_6 = u_2$.

(a) Write down the generator matrix and parity-check matrix for code I.

(b) Write out a decoding table for code I, assuming a BSC with crossover probability $\epsilon < \frac{1}{2}$.

(c) Given an exact expression for the probability of decoding error for code I and for code II. Which is larger?

(d) Find d_{min} for code I and code II.

(e) Give a counterexample to the conjecture that if one (N, L) parity check code has a larger minimum distance that another, then it has a smaller probability of error on a BSC. **HINT:** remember that p_e is an AVERAGE quantity whereas d_{min} is an extremal quantity.

4. Consider two parity-check codes of the same block length. Interpret the set of code words in each code as a vector-space under modulo-2 addition. Show that the set of sequences that are code words in both codes forms a subspace of each of the above vector spaces. Interpreting the sequences in this subspace as a parity-check code, describe a procedure for finding a parity-check matrix for the new code in terms of the parity-check matrices for the original codes.

5. Show that if a parity-check code has odd minimum weight, adding a parity-check digit that is a check on every digit in the code (modulo-2 sum all digits) increases the minimum weight by 1.