Rutgers University
The State University Of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

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Problem Set 4

1. You are approached by Shady Investment Corporation with an investment scheme. You are to give them C dollars on day 1. At the end of each day they will multiply the total money in your account by a random variable x and deposit the result in your account. Thus, at the end of day 1 you will have Cx_1 dollars in your account and this total amount will be used in the scheme on day 2. The distribution of x is:

$$p(x) = \begin{cases} \frac{1}{2} & x = 2 \\ \frac{1}{4} & x = \frac{1}{2} \\ \frac{1}{4} & x = \frac{1}{4} \end{cases}$$

   (a) Let Q_n be the amount of money in the account on day n. Please provide an analytic expression for Q_n in terms of C, and the appropriate x_i. What is the entropy in bits of the distribution $p(x)$?

   (b) Please calculate the expected value of Q_n. What is the entropy of Q_n? You may assume that the x_i are independent. What is the value of $\lim_{n\to\infty} E[Q_n]$? What does this seem to say about the effectiveness of the investment scheme?

   (c) Please calculate the expected value of $\log(Q_n)$. What is the entropy of the random variable $R_n = \log(Q_n)$? You may assume that the x_i are independent. What is the value of $lim_{n\to\infty} E[log(Q_n)]$? What does this seem to say about the effectiveness of the investment scheme?

   (d) You have uncovered a seeming paradox in the previous two parts. Is the investment scheme a good one or a bad one? You must quantify your argument. Does your answer change if the x_i are identically distributed as before but neither completely independent nor completely dependent? (HINT: Remember the central limit theorem).

2. You have just been solicited by Sadistic Coding Enterprises at a ridiculously high salary to design a block code for an i.i.d. binary source. You must provide a binary code of rate $R = (log M)/n$ where n is the number of consecutive source letters coded and M is the total number of code words used. Your would-be employers have set the value of R but you are free to choose n.

   (a) How many n-digit source sequences are there? For $R = 0.5$ what is M?

   In part (a) you saw that there may not be enough codewords for each possible source sequence; i.e., there is the possibility of being given a source sequence for which there is no codeword (block code default) because you don't have enough codewords to map to each possible source sequence. Thus, you must be selective in which source sequences you choose to code. You must be especially selective since your employment contract has a clause stating: *block code default $\rightarrow$ loss of favorite body part.*

   (b) Let the source letter probabilities be $p_0 = 0.5$ and $p_1 = 0.5$. Let $R = 0.5$. Can an n and M be found such that the probability of your losing a body part is infinitesimally small? Why/Why not?

(c) Suppose, $R = H(source) + \epsilon$ where ϵ is some positive number. Is it possible to find n and M such that the probability of your losing a body part is infinitesimally small? Why/Why not?

(d) For R as is part (c), please identify a method of choosing source sequences so that the probability of default can be made as small as possible. Be very explicit. Assume the source distribution is known.

(e) How would you change your answer in part (d) if the source is a Markov source? Once again you must be explicit.

3. You are given a source probability distribution with four elements $p_1 \geq p_2 \geq p_3 \geq p_4$ corresponding to source letters a_1, a_2, a_3 and a_4. A Huffman code using a binary code alphabet is desired for this source.

   (a) What is the minimum value of p_1 which guarantees that the source letter a_1 has a codeword 1 bit long?

   (b) What is the maximum value of p_1 which guarantees that the source letter a_1 has a codeword longer than 1 bit?

4. We know that $H(X|Y) \leq H(X)$. However, is $H(X|y) \leq H(X)$ for all experimental values of y?

5. Consider n (in general different) Discrete Memoryless Channels (DMCs) with capacities C_1, C_2 etc. The associated "sum" channel is that channel whose input and output alphabets are the unions of those of the original channels; i.e., the sum channel has all n channels available for use but only on channel may be used at a time. Show that the capacity of the sum channel is given by

$$C = \log_2 \sum_{i=1}^{n} 2^{C_i}$$

and find q_i, the probability of using the i^{th} channel. Interpret C as the average of the capacities of the individual channels plus the information conveyed by the selection of a channel. HINT: write the input distribution as $q(i)p_i(j)$ where $p_i(j)$ is the distribution on the inputs of the i^{th} channel given that the i^{th} channel is used. Then use the following theorem:

A probability distribution p on the channel input achieves capacity if and only if there exists a number C such that for all $p_j > 0$

$$\sum_k Q_{k|j} \log(Q_{k|j}/q_k) = C$$

and for $p_j = 0$

$$\sum_k Q_{k|j} \log(Q_{k|j}/q_k) \leq C$$

where q_k is the probability distribution of the output and $Q_{k|j}$ is the channel transition matrix.