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Problem Set 1 Solutions

1. None of the 6 possible implications is true. In fact, for each of the 8 possible combinations in which each of I, II and III holds or doesn't, it is possible to find a probability assignment satisfying the requirements of that combination. For example

$$\begin{array}{cc} ABC & \frac{1}{16} \\ AB\bar{C} & 0 \\ A\bar{B}C & \frac{1}{16} \\ A\bar{B}\bar{C} & \frac{2}{16} \\ \bar{A}BC & \frac{1}{16} \\ \bar{A}B\bar{C} & \frac{3}{16} \\ \bar{A}\bar{B}C & \frac{1}{16} \\ \bar{A}\bar{B}\bar{C} & \frac{7}{16} \end{array}$$

So, $P(A) = 1/4$, $P(B) = 5/16$ and $P(AB) = 1/16 \neq P(A)P(B)$. However, we have $P(A|C) = 1/2$, $P(B|C) = 1/2$ and $P(AB|C) = 1/4 = P(A|C)P(B|C)$. as well as $P(A|\bar{C}) = 1/6$, $P(B|\bar{C}) = 1/4$ and $P(AB|\bar{C}) = 0 \neq P(A|\bar{C})P(B|\bar{C})$. So II is true but neither I nor III is true.

One can futz around with the probability entries and show that none of the other possible implications is true either

2. (a) Trivial owing to the linearity of the expression for expectation.

   (b) Let $p_{XY}(x, y)$ be the joint density of X and Y. Then

$$E[XY] = \sum_{x,y} xyp_{XY}(x, y) = \sum_{x,y} xyp_X(x)p_Y(y) = E[X]E[Y]$$

   However, let $p_{XY}(1, 0) = p_{XY}(0, 1) = p_{XY}(-1, 0) = p_{XY}(0, -1) = 1/4$ we have $E[X] = E[Y] = 0 = E[XY]$. However, we also have $P(X = 1) = 1/4$ and $P(Y = 0) = 1/2$, but $P(X = 1, Y = 0) = 1/4 \neq P(X = 1)P(Y = 0)$. Thus uncorrelated does not imply independence.

   (c) $E[(X - m_x)(Y - m_y)] = E[XY] - m_x m_y$ which if uncorrelated is zero. Thus, $E[((X + Y) - (m_x + m_y))^2] = E[(X - m_x)^2 + (Y - m_y)^2 - 2E[(X - m_x)(Y - m_y)]$ which is just $E[(X - m_x)^2 + (Y - m_y)^2 = \sigma_x^2 + \sigma_y^2$ if X and Y are uncorrelated.

3. (a)

$$\sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} p_x(k) = \begin{array}{l} p_x(1) + p_x(2) + p_x(3) + \ldots \\ p_x(2) + p_x(3) + p_x(4) + \ldots \\ p_x(3) + p_x(4) + p_x(5) + \ldots \\ \ldots \end{array}$$

   Notice that $p_x(k)$ appears k times in the implied table. Thus

$$\sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} p_x(k) = \sum_{j=0}^{\infty} jp_x(j) = E[X]$$

(b) By analogy

$$E[X] = \int_0^\infty \mathrm{Prob}(x \geq t)dt$$

4. (a)

$$E[g(x)] = \int_0^\infty g p_g(g) dg = \int_0^t g p_g(g) dg + \int_t^\infty g p_g(g) dg$$

Since $g \geq t$ in the second integral we can bound the second integral as

$$\int_t^\infty g p_g(g) dg \geq t \int_t^\infty p_g(g) dg = t\mathrm{Prob}(g(x) \geq t)$$

Thus,

$$E[g(x)]/t \geq \mathrm{Prob}(g(x) \geq t)$$

(b) Let $g(x) = |x|$ and $t = \sigma E[X]$. We then have $E[|X|] > \sigma E[X]\mathrm{Prob}(|X| > \sigma E[X])$. Rewriting yields $\mathrm{Prob}(|x| > \sigma E[X]) \leq E[|X|]/(\sigma E[X])$. But since X is nonnegative, $E[X] = E[|X|]$ so we have

$$\mathrm{Prob}(x > \sigma E[X]) \leq 1/\sigma$$

the desired result.

As for humans, we have $E[X] = 5$ and the probability that someone is greater than 10 feet tall as 0.5 ... a pretty weak bound.

(c) A random variable is a variable whose value is determined by a mapping from a probability event space to some value space. Note that the value space need not simply be the reals; it can be a complex vector for example. However, usually it's just the reals, $\Re^1$.