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Problem Set 1

1. Let A, B, C be three chance events, i.e., subsets of a probability space. Consider the following possibilities.

   I A and B are independent.

   II A and B are conditionally independent given C.

   III A and B are conditionally independent given $\overline{C}$, where $\overline{C}$ is the complement of C.

   (a) Prove or find a counterexample (i.e., find a probability assignment for A, B, C and their intersections and complements) to the following statements: i) $I \rightarrow II$; ii) $II \rightarrow I$.

   (b) Of the remaining possible implications, indicate which ones are true and which ones aren't.

2. Let x and y be discrete random variables.

   (a) Prove that the sum of the expectations, $\overline{x} + \overline{y}$ is equal to the expectation of the sum, $\overline{x+y}$.

   (b) Prove that if x and y are statistically independent, then x and y are also uncorrelated (by definition, x and y are uncorrelated if $\overline{xy} = \overline{x}\ \overline{y}$). Find an example in which x and y are uncorrelated but still statistically dependent; i.e., show that statistical independence and lack of correlation are not equivalent statements about a pair of random variables.

   (c) Prove that if x and y are statistically independent, then the variance of the sum is equal to the sum of the variances. Is this relationship valid if x and y are uncorrelated but not statistically independent?

3. Let x be a nonnegative integer-valued random variable. Prove that

   (a)
$$E(x) = \sum_{j=0}^{\infty} Prob[x > j]$$

   (b) What is the analogous formula if x is a nonnegative continuous random variable?

4. Let $g(\cdot)$ be a nonnegative function and let x be a random variable.

   (a) Prove that $Prob[g(x) \geq t] \leq E[g(x)]/t$. You may choose either the discrete or continuous case. NOTE: if $g(x) = |x - E[x]|^2$ we obtain the Chebyshev inequality.

   (b) Use (a) to show that if x is a nonnegative random variable, then $Prob[x \geq tE[x]] \leq 1/t$. To see how weak this bound can be, suppose x represents the height of human beings and that $E[x] = 5$ft. What is the bound on the probability that someone is 10 feet tall? Were this bound tight, how often would you expect to see a 10 foot tall person?

   (c) What is a random variable?